Canonical trace radical property: a new property of commutative rings

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In this talk, all rings and algebras are assumed to be commutative with identity element and Noetherian. We present in this talk a new concept of Noetherian commutative rings called canonical trace radical property which stands between Gorenstein and Cohen-Macaulay properties. See [Miy] for details.

In order to explain this property, we first recall the concept of the trace of a module.

Definition 1 Let R be a ring and M an R-module. The trace of M, denoted by $\operatorname{tr}_R(M)$, is by definition

$$\operatorname{tr}_R(M) = \sum_{\varphi \in \operatorname{Hom}_R(M,R)} \varphi(M).$$

If R is clear from the context, we denote $\operatorname{tr}_R(M)$ by $\operatorname{tr}(M)$ for simplicity. Note that $\operatorname{tr}_R(M)$ is an ideal of R. Further, we see the following.

Remark 2 Let R and M be as above. Then $\operatorname{tr}_R(M)$ is the image of the natural map

$$\operatorname{Hom}_R(M,R) \otimes_R M \to R, \quad f \otimes m \mapsto f(m).$$

Thus, if S is a flat R-algebra and M is a finitely generated R-module, then $\operatorname{tr}_S(M \otimes_R S) = \operatorname{tr}_R(M)S$. In particular, if $\mathfrak{p} \in \operatorname{Spec}(R)$, then $\operatorname{tr}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{tr}_R(M)_\mathfrak{p}$.

Now we state the following.

Definition 3 Let R be a Cohen-Macaulay ring such that for any $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_\mathfrak{p}$ admits a canonical module $\omega_{R_\mathfrak{p}}$. If $\operatorname{tr}_{R_\mathfrak{p}}(\omega_{R_\mathfrak{p}})$ is a radical ideal for any $\mathfrak{p} \in \operatorname{Spec}(R)$, we say that R is a canonical trace radical ring (CTR ring for short).

Since the unit ideal is a radical ideal, Gorenstein rings are CTR rings. It is clear from the definition and Remark 2, the following.

Proposition 4 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module ω. Then R is CTR if and only if $\operatorname{tr}_R(\omega)$ is a radical ideal.*

By the description of the trace ideal by Herzog, Hibi and Stamate [HHS], we also see the following.

Proposition 5 *Let $R = \bigoplus_{n \in \mathbb{N}} R_n$ be an $\mathbb{N}$-graded Cohen-Macaulay ring with R_0 a field. Also, let ω be the graded canonical module of R. Then R is CTR if and only if $\operatorname{tr}_R(\omega)$ is a radical ideal.*

Next we state classes of rings which motivated us to define CTR property. The first one is the Schubert cycles. The homogeneous coordinate ring of a Schubert subvariety of a Grassmannian is called a Schubert cycle.

Let R be a Schubert cycle. Then R is a normal integral domain and according to the notation of [BV, §8], there is a special element $\epsilon \in R$ with the following property. If we denote the minimal prime ideals of (ϵ) by $P_0, P_1, \ldots, P_t$, then the divisor class group $\mathrm{Cl}(R)$ of R is generated by classes $\mathrm{cl}(P_0), \mathrm{cl}(P_1), \ldots, \mathrm{cl}(P_t)$ and the relation between $\mathrm{cl}(P_0), \mathrm{cl}(P_1), \ldots, \mathrm{cl}(P_t)$ in $\mathrm{Cl}(R)$ of R is $\sum_{i=0}^{t} \mathrm{cl}(P_i) = 0$ only. Express the canonical class, the divisor class in $\mathrm{Cl}(R)$ of the canonical module, as $\sum_{i=0}^{t} \kappa_i \mathrm{cl}(P_i)$, set $\kappa = \max\{\kappa_i \mid 0 \leq i \leq t\}$ and $\kappa' = \min\{\kappa_i \mid 0 \leq i \leq t\}$. Then, while κ and κ' depend on the representation of the canonical class, $\kappa - \kappa'$ is independent of the representation.

Fact 6 (Bruns-Vettter) *R is Gorenstein if and only if $\kappa - \kappa' = 0$.*

As for CTR property, we have the following.

Theorem 7 *R is CTR if and only if $\kappa - \kappa' \leq 1$.*

Next, we consider the Ehrhart ring of the stable set polytope of a graph. First we consider a cycle graph, that is, a graph consisting of one cycle only.

Let R be the Ehrhart ring of the stable set polytope of a cycle graph of length n. Then it is known the following.

Fact 8 (Hibi-Tsuchiya) *R is Gorenstein if and only if n is even or $n \leq 5$.*

We have the following.

Theorem 9 *R is CTR if and only if n is even or $n \leq 7$.*

Next, we consider perfect graphs. Let $G = (V, E)$ be a perfect graph. Set $k = \max\{\#K \mid K$ is a maximal clique of $G\}$ and $k' = \min\{\#K \mid K$ is a maximal clique of $G\}$. Also, let R be the Ehrhart ring of the stable set polytope of G. About Gorenstein property, the following is known.

Fact 10 (Ohsugi-Hibi) *R is Gorenstein if and only if $k - k' = 0$.*

We have the following fact.

Proposition 11 *If R is CTR, then $k - k' \leq 1$.*

$k - k' \leq 1$ is not sufficient for R to be CTR. However, in the case where G is a comparability graph of a poset, i.e. V is a poset and $E = \{\{x, y\} \mid x < y \text{ in } V\}$, we have the following.

Theorem 12 *Suppose that G is a comparability graph of a poset and R is level* or *anti-canonical level. Then R is CTR if and only if $k - k' \leq 1$.*

References

[BV] Bruns, W. and Vetter, U.: Determinantal rings. GTM 1327. Springer, (1988).

[HHS] Herzog, J., Hibi, T. and Stamate, D. I.: “The trace of the canonical module.” Isr. J. Math., 233 (2019), 133–165.

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